Int. J. Heat Mass Transfer. Vol. 30, No. 5, pp. 949–957, 1987
Printed in Great Britain

0017-9310/87 $3.00+0.00

A multi-zonal integral method for problems involving unsteady one-dimensional heat conduction with change of phase

PETER W. CARPENTER
Department of Engineering Science, University of Exeter, Exeter EX4 4QF, U.K.

(*Received* 14 *October* 1985 *and in final form* 22 *September* 1986)

Abstract—A method is presented for calculating the temperature in a solidifying slab of molten material with greatly differing cooling rates on the inner and outer surfaces. For the purposes of analysis the slab is divided into a number of zones having time-varying boundaries, e.g. solidification or conduction fronts. Appropriate approximations are made for the temperature profile and the heat-balance integral equation applied separately in each zone. This leads to systems of first-order ordinary differential equations which can be integrated numerically by using simple standard methods. Results are presented which correspond to the industrial casting of polypropylene film.

1. INTRODUCTION

THE WORK described in this paper was motivated by a requirement to predict temperature profiles in a polypropylene film during casting. The casting process involves passing molten film over a series of chill rollers which cools and solidifies it. Additional cooling, due to convection from the outer surface of the film, also occurs. The convective cooling could be boosted ten-fold by installing an air chamber. An assessment had to be made of the effect of this on the overall cooling process. In particular, predictions of the temperature profiles at the end of the first roller, with and without an air chamber, were required. To a good approximation the cooling process could be regarded as one-dimensional. Thus a method was required for calculating the temperature in a solidifying slab of initially molten material with greatly differing cooling rates on the inner and outer surfaces.

As is common in such industrial problems only approximate values were available for the various input parameters, such as the convective heat-transfer coefficients and the physical properties of the material. Consequently, provided any singular behaviour is properly treated, there would be little point in developing a highly accurate numerical approach to the problem. Thus the integral method described below was developed. This method is an extension to that devised by Goodman and others [1–4]. A singularity is found to occur when the liquid material first reaches its freezing temperature. A special procedure for dealing with this singularity is introduced. The method is fairly general and should be applicable to other similar situations. Several previous authors [1–10] have used integral methods to investigate problems similar to the present one. These authors, however, have considered model problems which are rather simpler than the present one. For instance, in most cases the initial temperature has been set equal to the melting temperature. This greatly simplifies matters since it means that the temperature of one of the phases remains constant, making any calculation thereof unnecessary. Yuen [8] includes initial sub-cooling in his analysis but in his case either fixed temperature or fixed heat-transfer rate is imposed as a boundary condition.

In the present method the heat-transfer domain is divided into zones. The boundaries of the zones are defined so as to have definite physical significance, e.g. a line of solidification or position of a conduction front. Appropriate approximations are made for the temperature profile in each zone and the heat-balance integral equation applied separately to each zone. In this way relatively simple systems of first-order ordinary differential equations are derived. These can be readily integrated numerically by simple standard techniques.

The paper is set out in the following way. The analytical treatment is developed in Section 2. Section 2.1 describes the cooling process and identifies three time periods—a sub-cooling period and two subsequent periods. The integral relations and boundary conditions are discussed in Section 2.2. The solution for Period I is obtained in Section 2.3. The temperature profiles and governing equations for Periods II and III are derived in Sections 2.4 and 2.5, respectively. Results and discussion are presented in Section 3. In Section 3.1 comparisons are made between the results of Westphal's [11] series solution and those obtained using the present method. However, Westphal's solution only applies to the relatively simple case where the initial temperature equals the freezing temper-

NOMENCLATURE

A	coefficient in quadratic approximation (6) for non-dimensional temperature
A_2, A_3	quantities defined below equations (12) and (19), respectively
B_2, B_3	quantities defined below equations (12) and (19), respectively
Bi_{0l}, Bi_{0s}	Biot number, h_0H/k, using $k = k_l$ and k_s, respectively
Bi_1	Biot number, h_1H/k_l
c	specific heat
C_2, $\tilde{C}_2$, C_3	quantities defined below equations (12), (17) and (19), respectively
D_3, E_3, F_3, G_3, H_3	quantities defined below equation (19)
h	heat-transfer coefficient
H	thickness of slab
k	thermal conductivity
L	enthalpy of solidification
N	number of zones
t	time
T	temperature
x	coordinate normal to surface measured from roller wall.

Greek symbols

α	thermal diffusivity
β	Stefan number, $L/\{c_l(T_i - T_r)\}$
$\delta_i(i = 1,2,3,4)$	location of zone boundaries
$\eta_i(i = 1,2,3,4)$	non-dimensional x-wise co-ordinate, $(\xi - \delta_{i-1})/(\delta_i - \delta_{i-1})$
θ	non-dimensional temperature, $(T - T_r)/(T_i - T_r)$
ν	parameter, 1 for the liquid phase, α_s/α_l for the solid phase
ξ	non-dimensional x-wise coordinate, x/H
ρ	density
τ	non-dimensional time, $\alpha_l t/H^2$.

Subscripts

0	inner surface
1	outer surface
a	ambient conditions
f	conditions at solidification front
i	initial conditions at $\tau = 0$
l	liquid phase
r	conditions inside roller (inner wall)
s	solid phase
w	conditions at surface
ξ	denotes differentiation with respect to ξ.

ature. In Section 3.2 the full version of the present method is applied to the industrial casting of polypropylene by way of an illustrative example. Finally, Section 4 contains a brief conclusion.

2. ANALYSIS

2.1. *Description of the cooling process*

The cooling and solidification of the film proceeds in a series of distinct phases. Since the initial temperature of the melt is well above the freezing point, T_f, there is an initial pre-cooling period (Period I) during which the temperature falls until the inner surface temperature reaches T_f. During Period I conduction fronts propagate in from the inner and outer surfaces at different rates. When the inner surface temperature has fallen to T_f, a solidification front begins to propagate into the molten slab (Period II). Throughout Period II the two conduction fronts continue to propagate into the molten slab from both the inner and outer surfaces. Period III begins when the conduction fronts meet. Until this point some part of the interior remains at the initial temperature, T_i.

If the cooling process continues for long enough on the first roller, the outer surface temperature will eventually fall to T_f. From this point a second solidification front begins to propagate in from the outer surface (Period IV). In the present work calculations were discontinued at the end of Period III since, in the particular industrial application under consideration, the end of the first roller was reached during Period III.

2.2. *Integral relations and boundary conditions*

Let the region between $x = 0$ (at the roller surface) and $x = H$ (the outer surface of the film) be divided into N zones. The zones will be so defined as to have definite physical significance. For example, during Period III zone 1 comprises the solidified material between $x = 0$ and $\delta_1 H$ (the solidification front); zone 2 comprises the molten material between $x = \delta_1 H$ and $\delta_2 H$ (the point of maximum temperature); and zone 3 lies between $x = \delta_2 H$ and $\delta_3 H = H$. The required integral relations are obtained by integrating the non-dimensional heat-conduction equation with respect to ξ ($= x/H$) across each zone. This gives

$$\nu\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\delta_{i-1}}^{\delta_i}\theta\,\mathrm{d}\xi - \theta(\delta_i)\frac{\mathrm{d}\delta_i}{\mathrm{d}\tau} + \theta(\delta_{i-1})\frac{\mathrm{d}\delta_{i-1}}{\mathrm{d}\tau}\right] = \theta_\xi(\delta_i) - \theta_\xi(\delta_{i-1}), \qquad i = 1,\ldots,N \tag{1}$$

where $\theta = (T - T_r)/(T_i - T_r)$, T and T_r are the local temperature and the temperature in the roller interior, respectively, τ is non-dimensional time defined as $\alpha_l t/H^2$ (α_l being the thermal diffusivity of the liquid phase), $\nu = 1$ for the liquid phase and $\nu = \alpha_s/\alpha_l$ for the solid phase (α_s being the thermal diffusivity of the solid phase), and $\theta_\xi = \partial\theta/\partial\xi$.

At the inner surface, $x = 0$, the boundary condition is

$$\theta_\xi = Bi_0\,\theta \qquad \text{at } \xi = 0 \tag{2}$$

where the Biot number, $Bi_0 \equiv h_0 H/k$, h_0 is the thermal conductance per unit area of the roller wall, $k = k_l$ (the thermal conductivity in the liquid phase during Period I) and $k = k_s$ (the thermal conductivity in the solid phase) subsequently.

At the outer surface, $x = H$, the boundary condition is

$$\theta_\xi = -Bi_1(\theta + \theta_a) \qquad \text{at } \xi = 1 \tag{3}$$

where $Bi_1 \equiv h_1 H/k_l$, h_1 is the convective heat-transfer coefficient, and θ_a is the non-dimensional temperature difference between the roller interior and the surroundings.

At the solidification front $\xi = \delta_1$ an energy balance gives

$$\theta = \theta_f \tag{4a}$$

$$k_s(\theta_\xi)_s - k_l(\theta_\xi)_l = (k_l/\beta)\,\mathrm{d}\delta_1/\mathrm{d}\tau \tag{4b}$$

where $\beta \equiv L/\{c_l(T_i - T_r)\}$ is in the form of a Stefan number and where suffixes s and l denote the solid and liquid phases, respectively, L is the enthalpy of solidification, c is the specific heat and $\theta_f \equiv (T_f - T_r)/(T_i - T_r)$.

At a conduction front it is assumed that

$$\theta_\xi = 0 \quad \text{and} \quad \theta = \theta_m \equiv (T_m - T_r)/(T_i - T_r) \tag{5}$$

where T_m is the initial temperature during Periods I and II but the maximum temperature within the film during Period III.

Within each zone the temperature profile is approximated by a quadratic, i.e.

$$\theta = A + B\eta_i + C\eta_i^2;$$
$$\eta_i \equiv (\xi - \delta_{i-1})/(\delta_i - \delta_{i-1}). \tag{6}$$

Coefficients A, B and C are determined by the appropriate combination of equations (2), (3), (4a) and (5). Equation (6) is then substituted into equation (1) to give N integral relations which together with equation (4b), when appropriate, give $N + 1$ first-order ordinary differential equations for the $N + 1$ unknown time-dependent parameters, $\delta_1,\ldots,\delta_{N-1}$, θ_w (θ at $\xi = 0$) and θ_m (during Period III only).

Superficially, perhaps, the present method resembles that due to Bell [7] in that he also split the domain into a number of strips or zones. Bell, however, used isotherms as the zone boundaries with the melt line as a special case. This technique would probably be difficult to use on the present problem in which the slab has a uniform initial temperature and develops large temperature gradients by the end of Period III.

2.3. *Solution for Period I*

The temperature profile in the inner conduction zone is obtained from equation (6) with boundary conditions (2) and (5) used to determine coefficients A, B and C. Thus

$$\theta = \frac{1}{1 + \frac{1}{2}\delta_1\,Bi_{01}}\left[1 + \delta_1\,Bi_{01}\,\eta_1 - \frac{1}{2}\delta_1\,Bi_{01}\,\eta_1^2\right] \tag{7}$$

where $\eta_1 = \xi/\delta_1$.

With substitution of equation (7), equation (1) takes the form

$$\left[\frac{4 + \delta_1\,Bi_{01}}{2 + \delta_1\,Bi_{01}}\right]\delta_1\,Bi_{01}\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta_1\,Bi_{01}) = 6Bi_{01}^2. \tag{8}$$

Equation (8) can be readily solved to give the inverse relationship

$$\tau = \frac{1}{12}\delta_1^2 + \frac{1}{3}\frac{\delta_1}{Bi_{01}} - \frac{2}{3Bi_{01}^2}\ln\left(1 + \frac{1}{2}\delta_1\,Bi_{01}\right). \tag{9}$$

A similar solution can be found for the outer conduction zone. Equation (9) agrees to well within 1% error with numerical solutions [12] for the same problem when Bi_{01} ranges from 0.1 to 10.0.

The value of δ_1 at the end of Period I and the beginning of Period II can be easily obtained from equation (7) by setting $\theta = \theta_f$ at $\eta_1 = 0$, giving

$$\delta_1 = (1 - \theta_f)\Big/\left(\frac{1}{2}Bi_{01}\,\theta_f\right). \tag{10}$$

This value can be substituted in equation (9) to obtain the corresponding value of τ.

2.4. *Temperature profiles and equations for Period II*

Boundary conditions (2), (3), (4a) and (5) are used, as appropriate, with equations (6) to obtain the following temperature profiles:

$$\theta = \theta_w\{1 + \delta_1\,Bi_{0s}(\eta_1 - \eta_1^2)\} + (\theta_f - \theta_w)\eta_1^2 \qquad : \text{Zone 1} \tag{11a}$$

$$\theta = \theta_f + (1 - \theta_f)(2\eta_2 - \eta_2^2) \qquad : \text{Zone 2} \tag{11b}$$

$$\theta = 1 \qquad : \text{Zone 3} \tag{11c}$$

$$\theta = 1 - \{(1 - \delta_3)Bi_1(1 + \theta_a)/(2 + Bi_1 - Bi_1\,\delta_3)\}\eta_4^2 \qquad : \text{Zone 4} \tag{11d}$$

where $Bi_{0s} \equiv h_0 H/k_s$.

Zone 1 comprises the solidified material and

$\eta_1 = \xi/\delta_1$ where δ_1 is the location of the solidification front. θ_w is the value of θ at $\xi = 0$. Zone 2 comprises the cooling molten material and extends from the solidification front at $\xi = \delta_1$ to the inner conduction front at $\xi = \delta_2$, so that $\eta_2 = (\xi - \delta_1)/(\delta_2 - \delta_1)$. Zone 3 comprises the molten material still remaining at the initial temperature and bounded by the inner and outer conduction fronts. Zone 4 comprises the molten material which is being cooled by convection from the outer surface; it extends from the outer conduction front at δ_3 to the outer surface $\xi = 1$ so that $\eta_4 = (\xi - \delta_3)/(1 - \delta_3)$.

The solution for δ_3, the location of the outer conduction front, can be found independently and takes the same form as for Period I. The remaining three unknown parameters, namely δ_1, δ_2 and θ_w are obtained by solving a system of three first-order ordinary differential equations derived from equations (1) and (4b). Substitution of equations (11a) and (11b) in equations (1) and (4b) leads to the following system of equations:

$$\dot{\theta}_w + A_2\dot{\delta}_1 = B_2 \qquad (12a)$$

$$2\dot{\delta}_1 + \dot{\delta}_2 = 6/(\delta_2 - \delta_1) \qquad (12b)$$

$$\dot{\delta}_1 = C_2 \qquad (12c)$$

where $(\dot{\ })$ denotes differentiation with respect to τ

$$A_2 = \frac{\theta_w\delta_1 Bi_{0s} - 2(\theta_f - \theta_w)}{\delta_1\left(2 + \frac{1}{2}\delta_1 Bi_{0s}\right)},$$

$$B_2 = \frac{6\dfrac{\alpha_l}{\alpha_s}\left(\dfrac{\theta_f - \theta_w}{\delta_1} - Bi_{0s}\theta_w\right)}{\delta_1\left(2 + \frac{1}{2}\delta_1 Bi_{0s}\right)}$$

and

$$C_2 = \beta\theta_f\left[\frac{k_s}{k_l}\left\{\frac{2(\theta_f - \theta_w)}{\delta_1\theta_f} - \frac{\theta_w Bi_{0s}}{\theta_f}\right\} - \frac{2}{\delta_2 - \delta_1}\frac{(1 - \theta_f)}{\theta_f}\right].$$

The system of equations (12) can be readily rearranged in a form suitable for numerical integration, i.e.

$$\dot{\delta}_1 = C_2, \qquad \dot{\delta}_2 = \frac{6}{\delta_2 - \delta_1} - 2C_2$$

and

$$\dot{\theta}_w = B_2 - A_2C_2. \qquad (13)$$

At the start of Period II the system of equations (13) is singular owing to the initial conditions $\delta_1 = 0$ and $\theta_w = \theta_f$; the initial values of δ_2 and τ are obtained from equation (10) (N.B. δ_2 in Period II corresponds to δ_1 in Period I). This difficulty is overcome by seeking solutions of the form

$$\delta_1 = a(\tau - \tau_i)^{\lambda}, \qquad \theta_f - \theta_w = b(\tau - \tau_i)^{\mu}$$

and

$$\delta_2 = \delta_{2i} \qquad (14)$$

in the vicinity of $\tau = \tau_i$ where τ_i and δ_{2i} are the initial values of τ and δ_2. This procedure follows the usual treatment of singularities in the theory of ordinary differential equations; see Ince [13] for example.

When solutions of the form of equations (14) are substituted into equations (12a) and (12b) and the limit $\bar{\tau} = (\tau - \tau_i) \rightarrow 0+$ is taken, they reduce to

$$-ab(\mu + \nu)\bar{\tau}^{\mu+\nu-1} + Aa^2\bar{\tau}^{2\nu-1} = B(b/a)\bar{\tau}^{\mu-\nu} - C$$

$$a^2\nu\bar{\tau}^{2\nu-1} = bD\bar{\tau}^{\mu} - Ea\bar{\tau}^{\nu}$$

where

$$A = \tfrac{1}{2}\theta_f Bi_{0s}, \qquad B = 3\alpha_l/\alpha_s, \qquad C = Bi_{0s}\theta_f B$$

$$D = 2(k_s/k_l)/\beta$$

$$E = \frac{\theta_f}{\beta}\left\{\frac{k_s}{k_l}Bi_{0s} + \frac{2(1 - \theta_f)}{\delta_{2i}\theta_f}\right\}.$$

Acceptable values of a and b can only be found from a solution of the above pair of equations provided

$$\mu = \nu = 1.$$

All other, apparently possible, choices for μ and ν lead to either $a = 0$ or $b = 0$. With the values of μ and ν determined the equations may be solved for a and b and the initial values for the derivatives $\dot{\delta}_1$, $\dot{\delta}_2$ and $\dot{\theta}_w$ (N.B. $\dot{\delta}_1 = a$ and $\dot{\theta}_w = -b$) are found to be

$$\dot{\delta}_{1i} = \beta\theta_f\left[\frac{k_s}{k_l}Bi_{0s} - \frac{2(1 - \theta_f)}{\delta_{2i}\theta_f}\right] \qquad (15a)$$

$$\dot{\delta}_{2i} = \frac{6}{\delta_{2i}} - 2\beta\theta_f\left[\frac{k_s}{k_l}Bi_{0s} - \frac{2(1 - \theta_f)}{\delta_{2i}\theta_f}\right] \qquad (15b)$$

$$\dot{\theta}_w = -Bi_{0s}\beta\theta_f\left[\frac{k_s}{k_l}Bi_{0s} - \frac{2(1 - \theta_f)}{\delta_{2i}\theta_f}\right]. \qquad (15c)$$

An alternative approach for dealing with the singular initial conditions is to assume that at the start of Period II the solid phase is so thin that the heat conduction process is quasi-steady. This implies that the temperature profile is linear and that the thermal resistances of the roller wall and the solid phase are equal. In this way θ_w can be explicitly obtained in terms of δ_1, i.e.

$$\theta_w = \theta_f\left\{1 - \frac{\delta_1 Bi_{0s}}{\delta_1 Bi_{0s} + 1}\right\}. \qquad (16)$$

This removes the need for equations (12c) and $(13)_3$. Equations $(13)_1$ and $(13)_2$ can now be rewritten as

$$\dot{\delta}_1 = \tilde{C}_2, \qquad \dot{\delta}_2 = \frac{6}{\delta_2 - \delta_1} - 2\tilde{C}_2 \qquad (17)$$

where

$$\tilde{C}_2 = \beta\theta_f\left[\frac{k_s}{k_l}\frac{Bi_{0s}}{1+\delta_1 Bi_{0s}} - \frac{2(\theta_i-\theta_f)}{\theta_f(\delta_2-\delta_1)}\right].$$

In the limit as $(\tau - \tau_i) \to 0$ the expressions for $\dot{\delta}_{1i}$, $\dot{\delta}_{2i}$ and $\dot{\theta}_{wi}$ obtained using the quasi-steady approximation agree with those given in equations (15). Some authors, however, e.g. Hrycak [5] and Goodman [1] use the quasi-steady approximation throughout the cooling process. In Section 3 results obtained by use of this approximation are compared to those obtained with the full set of equations (13).

Many authors [1, 3, 8, 9] have used the so-called collocation approach whereby the heat conduction equation is used to eliminate the explicit appearance of δ_1 from boundary condition (4b). Note that this approach is not required in the present method for which boundary condition (4b) is used to supply one of the differential equations of systems (12) and (13).

2.5. *Temperature profiles and equations for Period III*

At the start of Period III the two conduction fronts meet so that δ_2 merges with δ_3. The temperature profiles are derived in a similar way to those given in equations (11) and take the form

$$\theta = \theta_w\{1 + \delta_1 Bi_{0s}(\eta_1 - \eta_1^2)\} + (\theta_f - \theta_w)\eta_1^2 \qquad : \text{Zone 1} \quad (18a)$$

$$\theta = \theta_f + (\theta_m - \theta_f)(2\eta_2 - \eta_2^2) \qquad : \text{Zone 2} \quad (18b)$$

$$\theta = \theta_m - \{(1-\delta_2)Bi_1(\theta_m + \theta_a)/(2 + Bi_1 - Bi_1\,\delta_2)\}\eta_3^2 \qquad : \text{Zone 3} \quad (18c)$$

where θ_m is the non-dimensional maximum temperature. Zones 1 and 2 are as for Period II except that $\xi = \delta_2$ is now the location of the temperature maximum rather than a conduction front. Zone 3 of Period II disappears and Zone 4 becomes Zone 3.

The four unknown parameters δ_1, δ_2, θ_w and θ_m are found by solving a system of four first-order ordinary differential equations derived from equations (1) and (4b). The equations are obtained by substituting equations (18) into equations (1) and (4b) and take the form

$$\dot{\theta}_w + A_3\dot{\delta}_1 = B_3 \qquad (19a)$$

$$2\dot{\delta}_1 + \dot{\delta}_2 - G_3\dot{\theta}_m = 6/(\delta_2 - \delta_1) \qquad (19b)$$

$$D_3\dot{\delta}_2 + E_3\dot{\theta}_m = -F_3 \qquad (19c)$$

$$\dot{\delta}_1 = C_3 \qquad (19d)$$

where $A_3 = A_2$, $B_3 = B_2$ and

$$C_3 = \beta\theta_f\left[\frac{k_s}{k_l}\left\{\frac{2(\theta_f-\theta_w)}{\delta_1\theta_f} - \frac{\theta_w Bi_{0s}}{\theta_f}\right\} - \frac{2(\theta_m-\theta_f)}{(\delta_2-\delta_1)-\theta_f}\right]$$

$$D_3 = \frac{Bi_1(\theta_m+\theta_a)(1-\delta_2)\{4+(1-2\delta_2)Bi_1\}}{6+3Bi_1(1-\delta_2)}$$

$$E_3 = 2(1-\delta_2)\left\{1+\frac{1}{3}(1-\delta_2)Bi_1\right\}$$

$$F_3 = 2Bi_1(\theta_m+\theta_a).$$

and

$$G_3 = 2(\delta_2-\delta_1)/(\theta_m-\theta_f).$$

The system of equations (19) can be readily rearranged in a form suitable for numerical integration, as follows:

$$\dot{\delta}_1 = C_3, \qquad \dot{\delta}_2 = H_3, \qquad \dot{\theta}_w = B_3 - A_3C_3$$

and

$$\dot{\theta}_m = -(F_3 + D_3H_3)/E_3 \qquad (20)$$

where

$$H_3 = \left\{\frac{6}{\delta_2-\delta_1} - 2C_3 - \frac{F_3G_3}{E_3}\right\}\Big/\left(1+\frac{D_3G_3}{E_3}\right).$$

The initial conditions are supplied by the solutions at the end of Period II.

3. RESULTS AND DISCUSSION

3.1. *Comparison with Westphal's series solution*

Before applying the present method to the particular industrial problem for which it was developed, comparisons will be made with Westphal's [11] series solution. Westphal considered the case where the initial temperature is at the freezing point. The heat-transfer domain is semi-infinite with a convective boundary condition of the form

$$-k\frac{\partial T}{\partial x} = h(T - T_a) \qquad \text{at } x = 0.$$

For the special case of the semi-infinite domain with $T_i = T_f$ and T_a constant it can be shown by elementary dimensional analysis that the solidification front, x_f, and surface temperature may be expressed as a function of two parameters

$$\frac{x_f h}{k} = fn\left(\beta, \frac{\alpha h^2 t}{k^2}\right)$$

$$\theta_w = fn\left(\beta, \frac{\alpha h^2 t}{k^2}\right).$$

Westphal presented his solutions for $x_f h/k$ and θ_w in the form of a series in powers of $\alpha h^2 t/k^2$.

Table 1. Comparison between the present theory and Westphal's theory

	hx_f/k			θ_w		
$\alpha\frac{h^2}{k^2}t$	Westphal	Truncation error	Carpenter	Westphal	Truncation error	Carpenter
	Case (a): ice, $\beta = 35.69$					
0.1	0.00280	0.15×10^{-7}	0.00280	0.997	0.24×10^{-6}	0.997
1.0	0.0276	0.16×10^{-4}	0.0276	0.973	0.33×10^{-3}	0.973
10.0	0.247	0.018	0.249	0.782	0.32	0.802
	Case (b): ice, $\beta = 8.923$					
0.1	0.00111	0.15×10^{-5}	0.00111	0.989	0.1×10^{-3}	0.989
1.0	0.106	0.15×10^{-2}	0.106	0.904	0.09	0.905
10.0	0.267	6	0.787	—	—	0.563
	Case (c): polyethylene, $\beta = 0.973$					
0.001	0.00103	0.12×10^{-7}	0.00103	0.999	0.12×10^{-5}	0.999
0.01	0.0102	0.12×10^{-4}	0.0102	0.990	0.13×10^{-3}	0.990
0.1	0.0939	0.013	0.0943	0.913	0.138	0.917
	Case (d): lead, $\beta = 0.622$					
0.001	0.00161	0.11×10^{-6}	0.00161	0.998	0.35×10^{-4}	0.998
0.01	0.0158	0.11×10^{-3}	0.0158	0.985	0.56×10^{-2}	0.985
0.1	0.130	0.135	0.138	0.853	0.465	0.886

For the special case presently being considered the initial solutions (15) reduce to

$$\dot{\delta}_1 = \frac{1}{\beta}\frac{k_s}{k_l}Bi_{0s} \quad \text{and} \quad \dot{\theta}_w = -Bi_{0s}\dot{\delta}_1$$

i.e.

$$\frac{x_f h_s}{k_s} = \frac{1}{\beta}\frac{\alpha_s h_s^2 t}{k_s^2}$$

$$\theta_w = 1 - \frac{1}{\beta}\frac{\alpha_s h_s^2 t}{k_s^2}.$$

These initial solutions are in exact agreement with the leading terms of Westphal's series for $x_f h/k$ and θ_w. Note, however, that in the limit as $t \to 0+$ the assumed temperature profile used in the present method for the solid phase is exact.

Further comparisons between the present integral method and Westphal's series solution are made in Table 1. Four cases, corresponding to real industrial applications, are considered giving a wide range of Stefan number, β. (Note that in the present context c_s is used in the definition of β.)

Case (a): ice. $T_f = 0°\mathrm{C}$, $T_a = -5°\mathrm{C}$,
$h = 10\,\mathrm{W\,m^{-1}\,K^{-1}}$, $L = 373\,\mathrm{kJ\,kg^{-1}}$,
$c = 2.09\,\mathrm{kJ\,kg^{-1}\,K^{-1}}$.

Case (b): ice. As above except $T_a = -20°\mathrm{C}$.

Case (c): polypropylene. $T_f = 120°\mathrm{C}$, $T_a = 20°\mathrm{C}$,
$h = 17.5\,\mathrm{W\,m^{-1}\,K^{-1}}$,
$L = 210\,\mathrm{kJ\,kg^{-1}}$,
$c = 2.16\,\mathrm{kJ\,kg^{-1}\,K^{-1}}$.

Case (d): lead. $T_f = 327°\mathrm{C}$, $T_a = 40°\mathrm{C}$,
$h = 120\,\mathrm{W\,m^{-1}\,K^{-1}}$,
$L = 24.7\,\mathrm{kJ\,kg^{-1}}$,
$c = 0.138\,\mathrm{kJ\,kg^{-1}\,K^{-1}}$.

The conditions for Case (d) were taken from Hills and Moore [14]. The truncation-error estimate given in Table 1 for Westphal's solution was made by dividing the last terms of the series given in ref. [11], i.e. the $O\{(\alpha h^2 t/k^2)^4\}$ terms, by the sums of the other given terms. It can be seen from Table 1 that, provided this truncation-error remains reasonably small, there is almost perfect agreement between the results obtained by Westphal and the present method. Note that although Westphal's solution is restricted to relatively small values of $\alpha h^2 t/k^2$, particularly when β is small, there is no such restriction on the present methods.

A search has been undertaken for published numerical solutions for comparison with the present method. However, no suitable examples were found. It is worth noting that, in order to obtain numerical solutions for the sort of problem considered in Section 3.2, a special procedure would in any case be required to deal with the singularity at the start of Period II and it is probable that this procedure would be analytic in form.

3.2. *Application of method to the casting of polypropylene*

For the particular industrial problem under consideration, involving the casting of polypropylene film, the following input parameters and material properties were specified.

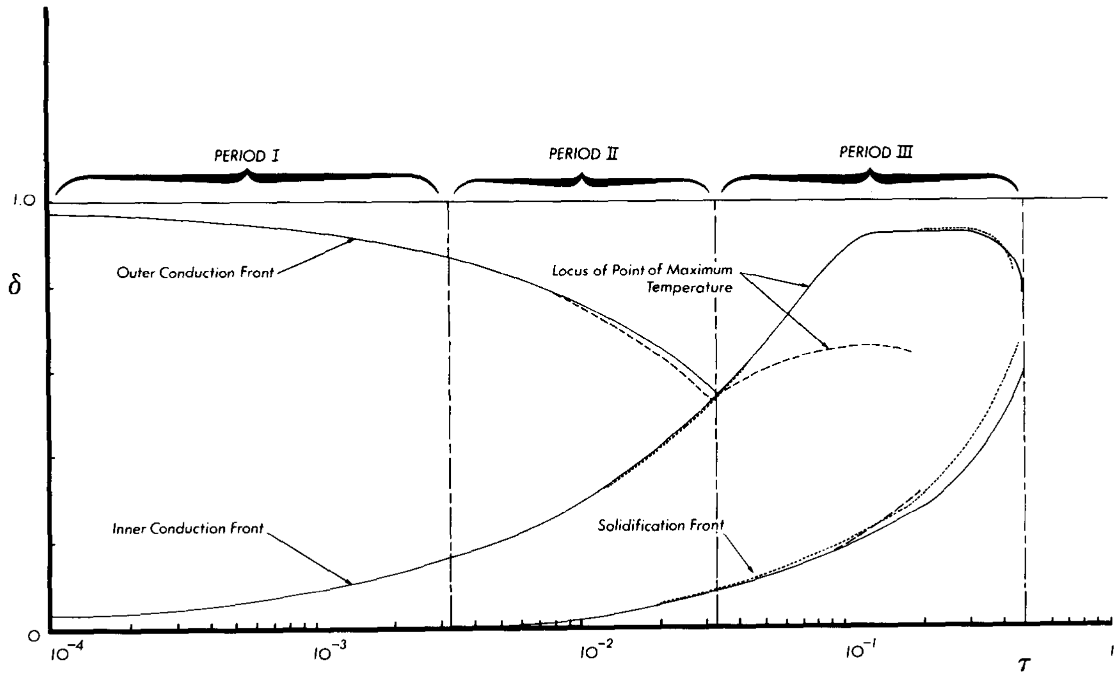

FIG. 1. The development of conduction and solidification fronts with non-dimensional time: ——, $Bi_1 = 0.2$; – – –, $Bi_1 = 2.0$; ·····, $Bi_1 = 0.2$ with quasi-steady approximation.

Initial thickness of polypropylene film = 1.35 mm.

3.2.1. *Temperatures.* $T_i = 280°\text{C}$, $T_r = 20°\text{C}$, $T_a = 20°\text{C}$, $T_f = 120°\text{C}$.

3.2.2. *Material properties at* 120°C. $\rho_l = 800\,\text{kg}\,\text{m}^{-3}$, $k_l = 0.012\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$, $c_l = 2.5\,\text{kJ}\,\text{kg}^{-1}\,\text{K}^{-1}$, $\alpha_l = 5.1 \times 10^{-4}\,\text{cm}^2\,\text{s}^{-1}$, $k_s/k_l = 1.425$, $\alpha_s/\alpha_l = 1.65$ and $L = 210\,\text{kJ}\,\text{kg}^{-1}$.

The variation of material properties with temperature was neglected in the present work.

3.2.3. *Heat-transfer coefficients.* The heat-transfer coefficient for the roller was approximately $h_0 = 1750\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$ corresponding to $Bi_{0l} = 20.0$. The values of the convective heat-transfer coefficient for the outer surface were taken as

Ambient air: $h_1 = 17.5\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$ corresponding to $Bi_1 = 0.2$
Air chamber: $h_1 = 175\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$ corresponding to $Bi_1 = 2.0$.

Typically the film takes about 3.9 s to reach the end of the first roller corresponding to a non-dimensional time of $\tau = 0.109$.

The systems of equations (13) and (20) were integrated numerically using a Runge–Kutta–Merson method but many other standard numerical methods would have been equally suitable.

The computed values of the locations of the conduction and solidification fronts are plotted in Fig. 1 for both cases of interest, namely $Bi_1 = 0.2$ and 2.0. The beginning and ends of the periods shown in Fig. 1 correspond to $Bi_1 = 0.2$. The corresponding values of non-dimensional temperatures (namely the inner and outer surface temperatures, θ_{w0} and θ_{w1}, and maximum temperature, θ_m) are plotted in Fig. 2. The computations were continued until the end of Period III, i.e. until θ_{w1} reached θ_f. For the industrial application in question temperature profiles were required at the end of the first roller corresponding to $\tau = 0.109$. These profiles are plotted in Fig. 3.

In all three figures results obtained with use of the quasi-steady approximation are also presented. As explained towards the end of Section 2.4 this approximation involves the use of a linear temperature profile in the solid phase and has the advantage of reducing the number of dependent variables by one. By comparing the quasi-steady results with the more accurate ones in Figs. 1–3 it can be seen that the quasi-steady approximation is reasonably accurate, especially during the earlier stages of the cooling process.

4. CONCLUSIONS

A simple integral method has been developed and shown to be capable of dealing with a relatively complex, transient, one-dimensional heat-conduction problem involving change of phase. The method is fairly flexible. For example, computations could be continued into Period IV or on to the next roller by introducing additional zones with the corresponding equations. The method has already been applied to another industrial problem [15].

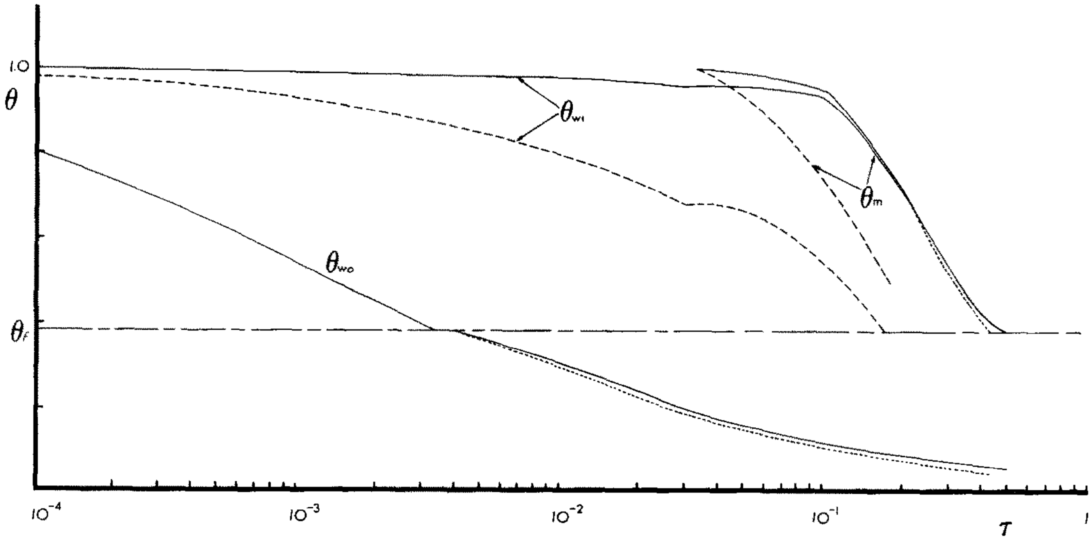

FIG. 2. Development of dimensionless maximum temperature and outer and inner wall temperatures with non-dimensional time (key as in Fig. 1).

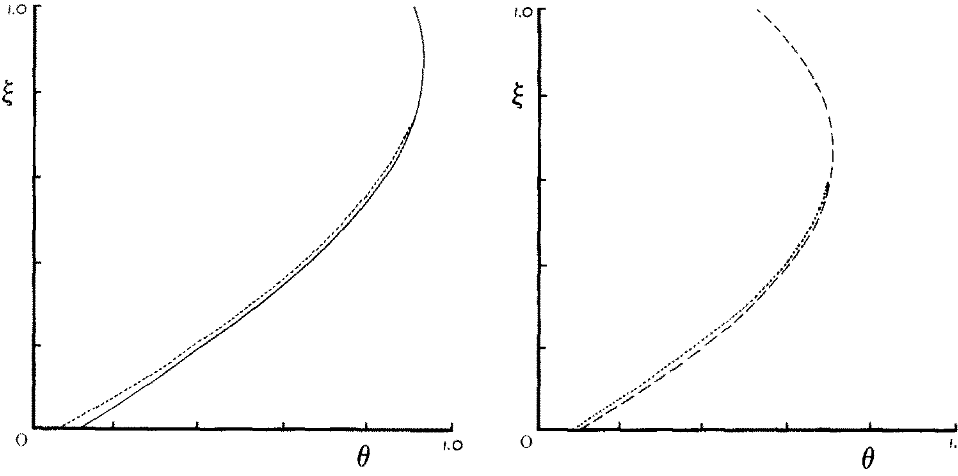

FIG. 3. Temperature profiles at $\tau = 0.109$: ·····, quasi-steady approximation.

REFERENCES

1. T. R. Goodman, The heat-balance integral and its application to problems involving a change of phase, *Trans. Am. Soc. mech. Engrs* **80**, 335–342 (1958).
2. T. R. Goodman and J. J. Shea, The melting of finite slabs, *Am. Soc. mech. Engrs J. Appl Mech.* **32**, 16–24 (1960).
3. G. Poots, On the application of integral methods to the solution of problems involving the solidification of liquids initially at fusion temperature, *Int. J. Heat Mass Transfer* **5**, 525–531 (1962).
4. T. R. Goodman, Applications of integral methods to transient nonlinear heat transfer, *Adv. Heat Transfer* **1**, 51–122 (1964).
5. P. Hrycak, Problem of solidification with Newton's cooling at the surface, *A.I.Ch.E. Jl* **9**, 585–589 (1963).
6. K. J. Mody and M. N. Özişik, Melting of a semi-infinite solid with nonlinear heat transfer at the boundary, *Lett. Heat Mass Transfer* **2**, 487–494 (1975).
7. G. E. Bell, A refinement of the heat balance integral method applied to a melting problem, *Int. J. Heat Mass Transfer* **21**, 1357–1362 (1978).
8. W. W. Yuen, Application of the heat balance integral to problems with initial sub-cooling, *Int. J. Heat Mass Transfer* **23**, 1157–1160 (1980).
9. B. Jayaram and W. Streider, An analysis of substrate heat losses in Stefan's problem with a constant flux, *Int. J. Heat Mass Transfer* **26**, 786–790 (1983).
10. T. W. Davies, The radiant cooling of a finite solid, *Int. Commun. Heat Mass Transfer* **11**, 311–321 (1984).
11. K. O. Westphal, Series solution of freezing problem with the fixed surface radiating into a medium of arbitrary varying temperature, *Int. J. Heat Mass Transfer* **10**, 195–205 (1967).
12. A. J. Chapman, *Heat Transfer*, 2nd edn. Macmillan, New York (1967).
13. E. L. Ince, *Ordinary Differential Equations*. Longmans, Green, London (1926).
14. A. W. D. Hills and M. R. Moore, The solidification of pure metals under unidirectional heat flow conditions. Part 1—solidification with zero superheat, *Trans. metall. Soc. A.I.M.E.* **245**, 1481–1492 (1969).
15. T. W. Davies and P. W. Carpenter, Transient conduction in a slab with chemical reaction, *Int. Commun. Heat Mass Transfer* **12**, 667–676 (1985).

UNE METHODE INTEGRALE MULTI-ZONALE POUR DES PROBLEMES DE CONDUCTION THERMIQUE VARIABLE; MONODIMENSIONNELLE; AVEC CHANGEMENT DE PHASE

Résumé—On présente une méthode pour calculer la température dans une couche de matériau fondu qui se solidifie avec des vitesses de refroidissement très différentes sur les surfaces interne ou externe. La couche est divisée en un nombre de zones ayant des frontières mobiles dans le temps, fronts de solidification ou de conduction. Des approximations appropriées sont faites pour le profil de température et l'équation intégrale du bilan de chaleur appliqués séparément dans chaque zone. Ceci conduit à des systèmes d'équations différentielles du premier ordre qui peuvent être intégrées numériquement en utilisant des méthodes classiques simples. Des résultats sont présentés qui correspondent à la production industrielle du film de propylène.

EIN MEHRSCHICHTEN-VERFAHREN ZUR LÖSUNG VON INSTATIONÄREN EINDIMENSIONALEN WÄRMELEITPROBLEMEN MIT PHASENWECHSEL

Zusammenfassung—Es wird eine Methode zur Berechnung der Temperatur in einer erstarrenden Schicht aus geschmolzenem Material vorgestellt, wobei die Abkühlungsgeschwindigkeiten an den inneren bzw. äußeren Oberflächen sehr unterschiedlich sind. Zum Zweck der Berechnung wurde die Schicht in eine Anzahl von Zonen mit zeitveränderlichen Berandungen, wie z.B. Erstarrungs- oder Erwärmungsfront, unterteilt. Es werden geeignete Näherungen für das Temperaturprofil und die Integralgleichung der Wärmestrombilanz durchgeführt und für jede Zone separat angewandt. Dies führt zu einem Gleichungssystem, bestehend aus gewöhnlichen Differentialgleichungen 1. Ordnung, die mit Hilfe einfacher Lösungsmethoden numerisch integriert werden können. Die Ergebnisse, die mit den Gegebenheiten bei der industriellen Gußerzeugung von Polypropylen-Folien übereinstimmen, werden vorgestellt.

МНОГОЗОНАЛЬНЫЙ ИНТЕГРАЛЬНЫЙ МЕТОД ДЛЯ НЕСТАЦИОНАРНЫХ ОДНОМЕРНЫХ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ С ФАЗОВЫМИ ПРЕВРАЩЕНИЯМИ

Аннотация—Предложен метод расчета температуры в затвердевающем слитке расплавленного материала, внутренняя и наружная поверхности которого охлаждаются с разной интенсивностью. При анализе слиток делился на несколько зон с подвижными границами, т.е. фронтами затвердевания или теплопроводности. Температурные профили аппроксимировались, а для каждой зоны в отдельности применялось интегральное уравнение теплового баланса. Этот подход приводит к системам обыкновенных дифференциальных уравнений первого порядка, которые интегрируются численного обычными методами. Расчеты соответствуют случаю промышленной отливки полипропиленовой пленки.